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# TIME RESOLUTION OF THE ST. PETERSBURG PARADOX: A REBUTTAL

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## Abstract

Peters (2011) claims to provide a resolution of the three century old St Petersburg paradox by using time averages and thereby avoiding the use of utility theory completely. Peters also claims to have found an error in Menger (1934, 1967) who established the vulnerability of any unbounded utility function to the St Petersburg paradox. This paper argues that both these claims in Peters (2011) are incorrect. The time average argument can be circumvented by using a single random number (between zero and one) to represent the entire infinite sequence of coin tosses, or alternatively by applying a time reversal to the coin tossing. Menger's proof can be reinstated by comparing the utility of playing the Super St Petersburg game to the utility of an arbitrarily large sure payoff.

## Introduction

The St Petersburg paradox invented by Nicolaus Bernoulli in 1713 is very simple: a fair coin is tossed until a head occurs. If the head occurs at the  $n$ 'th toss, the payoff is  $2^{n-1}$ . Note that with probability 1, a head would occur and the game would terminate.

Since the probability of the first head occurring at the  $n$ 'th toss is  $2^{-n}$ , the expected payoff is infinite:

$$\sum_{n=1}^{\infty} 2^{-n} 2^{n-1} = \left( \frac{1}{2} + \frac{1}{2} + \dots \right)$$

The paradox is that intuitively and in practice, this game does not have a large (let alone infinite) value – people are often willing to pay only a single digit price for participating in the game.

Daniel Bernoulli solved the paradox by arguing that we must compute not the expected payoff, but the expected utility. If the utility of wealth is a sufficiently concave function like the logarithm, then the expected utility of playing the St Petersburg game is finite and quite modest.

## Peters and the Kelly Principle

Peters (2011) claims to solve the paradox by using time averages instead of ensemble averages. His key result is

**Theorem 6.2.** The time-average exponential growth rate in the St Petersburg lottery is  $\bar{g} = \sum_{n=1}^{\infty} (1/2)^n \ln(w - c + 2^{n-1}) - \ln(w)$ .

Peters argues that “If the player can expect his wealth to grow over time, and he has no other constraints, then he should play the game; if he expects to lose money over time, then he should not play.” This is essentially the same as the Kelly (1956) criterion and Peters shows that the fair price is the same as that obtained by Daniel Bernoulli using logarithmic utility.

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## Reformulating the paradox without time

I argue that this resolution of the paradox is inadequate because it is trivial to reformulate the St Petersburg game without invoking a repeated coin tossing or any other sequential process at all. The St Petersburg game is equivalent to the following random number game.

- A random number lying between 0 and 1 is chosen.
- If the first non zero digit in the binary representation of this number is at the  $n$ 'th digit, the payoff is  $2^{n-1}$ .

Several quantum mechanical and other physical devices can be used for generating a random number uniformly distributed between 0 and 1. Given any random process with a known distribution, we can obtain a random number between 0 and 1 by using the well known result that if a random variable  $\tilde{X}$  has cumulative distribution function  $F(x)$ , then  $F(\tilde{X})$  is a random variable uniformly distributed over the unit interval.

For example, the decay of an unstable nucleus is entirely random and memoryless – the decay time follows the exponential distribution. If  $\tilde{t}$  is the (random) time at which the nucleus decays, the quantity  $1 - \exp(-\lambda\tilde{t})$  is a random number between 0 and 1 because the cumulative distribution function of the exponential distribution is  $(1 - \exp(-\lambda x))$ . Here  $\lambda$  is a physical constant associated with the given unstable nucleus.

Of course, physical constraints might limit the accuracy with which the time  $\tilde{t}$  can be measured, but mathematical discussions of the St Petersburg paradox (including that of Peters) abstract away from physical constraints. Introducing physical constraints eliminates the St Petersburg paradox too easily, and is therefore a mathematically and philosophically uninteresting route. For example,

- During an individual's lifetime only a finite number of tosses of the coin can take place.
- For large  $n$ , the promised payoff would not only exceed the wealth of the richest person in the world, but would also exceed total world wealth.

Any interesting discussion of the St Petersburg paradox must abstract away from such physical constraints and focus on the logical structure of the problem. In the same spirit, I also ignore physical constraints in the process of generating a random number lying between 0 and 1.

I would also add that the randomness of coin tossing is founded on the computational complexity of chaotic dynamics. Unlike genuinely random quantum phenomena (like the decay of a nucleus), coin tossing is not truly random (Diaconis et al., 2007).

## Reformulating the paradox with reversed time

In this section, I alter the St Petersburg game slightly to reverse the time while sticking to coin tossing. I define the Time Reversed St Petersburg Game as follows:

- An auxiliary coin is tossed until the number of heads exceeds the number of tails. Suppose this requires  $N$  tosses of the auxiliary coin.
- The main coin is tossed  $N$  times and the St Petersburg payoff formula is applied to the sequence of tosses in the *reverse order*. Specifically, if no head occurs while tossing the main coin, the payoff is 0, otherwise if the *last* head occur at the  $k$ 'th toss, the payoff is  $2^n$  where  $n \equiv N + 1 - k$  is the number of the "successful" toss counting from the end instead of the beginning.

It is not hard to show that the expected payoff from the Time Reversed St Petersburg Game is also infinite. First, we compute the conditional expectation of the payoff given that the main coin is tossed  $N$  times. When  $N$  is fixed, we have a finite version of the St Petersburg game and the expectation is a finite sum of  $N$  halves:

$$\sum_{n=1}^N 2^{-n} 2^{n-1} = \left( \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \right) = \frac{N}{2}$$

To show that the expectation of the whole game (including the auxiliary coin tossing) is infinite, it is sufficient to show that:

1. The tossing of the auxiliary coin will terminate with probability 1 so that the tossing of the main coin can begin; and
2. The expectation of  $N$  is infinite.

Both of these are standard results in the theory of waiting times for Bernoulli trials – for example, Feller (1950, Chapter XI.3).

Counting from the end instead of from the beginning completely invalidates Peters' solution of the paradox. In the Time Reversed St Petersburg Game there is no "growth over time"; there is only wild oscillation. Until the last coin is tossed, we do not know whether the payoff will be large or small. In the regular St Petersburg game, if no head has emerged after  $n$  tosses, it is certain that the payoff is at least  $2^n$ . Nothing similar can be said in the reversed sequence. If a head occurs early in the tossing of the main coin and this is followed by a long sequence of tails, a hope may arise of a large payoff, but the payoff can drop to a small amount (as low as 1) if a head emerges towards (at) the end. Even with a mere 10 tosses remaining, the probability of a large payoff is less than 0.1% (the probability of no head in the remaining 10 tosses is  $2^{-10}$ ).

If one desires only a large (but not infinite) expectation – much larger than the price that anybody would reasonably pay for the game, the Time Reversed St Petersburg Game can be dramatically simplified by eliminating the auxiliary coin and simply fixing  $N$  at some large number (say a million). The expected value of the game with this fixed  $N$  is half a million as compared to the single digit price that most people would pay for this game.

## Menger's Super St Petersburg Game

Menger (1934, 1967) introduced the Super St Petersburg Game which has infinite expected value even with a logarithmic (or any other unbounded) utility function. The core of Menger's idea is to increase the payoff dramatically from  $2^n$  to a linear function of  $\exp(2^n)$  so that we get  $2^n$  after applying the logarithmic utility function, and the utility increases by  $2^n$ . Just as the original St Petersburg Game has infinite expected value, the Super St Petersburg Game has infinite expected utility gain.

Peters (2011) argues correctly that this does not prove that a person would pay an infinite price for playing this game. An infinite price would lead to an infinite utility loss and subtraction of infinities can be quite slippery. Peters is however wrong in suggesting that Menger made a mathematical mistake – Menger never claimed that a person would pay an infinite price.

Samuelson (1977, footnote 4) explains the matter very succinctly:

Instead of asking what Paul will pay rather than not play the game, [Bernoulli and Menger] for the most part ask, "How much will Paul have to be given to make him as well off as if he received the game's positive gains."

Samuelson describes the difference mathematically as the difference between the roots  $f$  and  $f^*$  of the two equations:

$$U(W) = \sum 2^{-i}U(W - f + X_i)$$

and

$$U(W + f^*) = \sum 2^{-i}U(W + X_i)$$

where  $U$  is the utility function,  $W$  is the initial wealth (before playing the game) and  $X_i$  is the payoff if the first head occurs at the  $i$ 'th toss. In the first equation,  $f$  is subtracted from the wealth in every summand on the RHS because the player has to pay the price  $f$ . In the second equation,  $f^*$  is added to the wealth on the LHS because when the game is not played, the player receives  $f^*$ . Samuelson points out that Menger (1967, p. 222) recognizes that  $f < f^*$  for concave utility functions.

Menger is absolutely right that in the Super St Petersburg Game,  $f^*$  is infinite. A person would prefer the Super St Petersburg Game to an arbitrarily large certain prize. Menger's proof that any unbounded utility is susceptible to the St Petersburg paradox is mathematically impeccable.

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