An Introductory Note on Two Curve Discounting[1]

> *"LCH.Clearnet Ltd (LCH.Clearnet), which operates the world's leading interest rate swap (IRS) clearing service, SwapClear, is to begin using the overnight index swap (OIS) rate curves to discount its $218 trillion IRS portfolio. Previously, in line with market practice, the portfolio was discounted using LIBOR. ... After extensive consultation with market participants, LCH.Clearnet has decided to move to OIS to ensure the most accurate valuation of its portfolio for risk management purposes."*
>
> – LCH.Clearnet[2], June 17, 2010

> *"Ten years ago if you had suggested that a sophisticated investment bank did not know how to value a plain vanilla interest rate swap, people would have laughed at you. But that isn't too far from the case today."*
>
> – Deus Ex Machiatto[3], June 23, 2010

At what rate should we discount (or what is the risk free rate)?

1. **The old old model:** The risk free rate is the rate at which the government borrows. This model assumes that governments do not default and that government bond yields are not contaminated by liquidity or tax premiums

2. **The old model:** The risk free rate is the (Libor) rate at which the big international (Libor rated) banks borrow at short maturities (say three months). This model assumes that the credit spread over a short horizon (say three months) of a highly rated (say AA) bank is practically zero. This is justified in a Merton type model because short horizon default rates are zero for highly rated entities. In the Merton model, firms slide into default after repeated downgrades, and over short horizons, highly rated firms can experience downgrades and not defaults. In this model, the (TED[4]) spread between the government bond yield and Libor is a liquidity or tax premium and is not a credit spread. This model ignores "jump to default" risk.

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[1] This introductory note is my interpretation of the results presented in various papers listed in the bibliography and does not claim to contain any original content. I would like to acknowledge valuable comments received from Prof. Sidharth Sinha on earlier drafts of this note. Comments and suggestions are most welcome.

[2] http://www.lchclearnet.com/media_centre/press_releases/2010-06-17.asp

[3] http://blog.rivast.com/?p=3661

[4]TED stands for Treasury-EuroDollar and denotes the spread between the Treasury Bill yield and the Libor rate for the same maturity (usually three months).

3. **The new model:** The risk free rate is the OIS/Eonia[1] rate obtained by compounding the overnight interest rate at which Libor rated banks borrow. One way to justify this is to argue that the Merton style argument for zero credit risk at short horizons is valid for one day but not for longer periods. Another justification is that OIS/Eonia is the rate that is paid on cash collateral and is therefore the correct discounting rate for collateralized forward contracts and swaps. The latter argument does not require that Libor rated banks have zero credit risk at even overnight horizons.

Why did the global financial crisis change the idea of the risk free rate?

The Merton type argument that highly rated entities have zero credit risk at short horizons requires perfect observability of the balance sheet (asset values and debt). The Merton model states that default takes place when the market value of assets drops below the value of the debt (or more generally a default boundary that depends on the level of debt). If the asset value follows a diffusion process (without jumps), and the current value of assets is far above the default boundary, then default is practically impossible over a short time horizon because a diffusion takes time to move the asset value that far.

However, the crisis of 2008 showed that bank balance sheets are terribly opaque. Banks valued illiquid assets not by marking to market but by marking to model or sometimes marking to myth. Similarly, the true liabilities of the bank were hidden using various off balance sheet vehicles (for example, the ABCP[2] conduits and other SPV[3]s). This meant that the true distance to default[4] could be much lower than the estimated distance to default based on observable parameters. The unreliability of accounting information creates a "jump to default" risk and creates non negligible credit spreads at short time horizons[5]. Jumps in asset values are another explanation for "jump to default" risk but this does not appear to have been relevant for the banks during the global financial crisis[6].

The opaqueness of bank balance sheets and the speed with which some banks became distressed means that even one-month Libor can no longer be regarded as risk free. Whether overnight Libor can be regarded as risk free is an open question.

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[1]OIS (Overnight Index Swap) is the term used in the US while Eonia (European OverNight Index Average) is the term used in Europe.

[2]Asset backed commercial paper

[3]Special Purpose Vehicles

[4]The gap between the current asset value and the default boundary measured in standard deviations of the asset price diffusion.

[5]See Duffie and Lando (2001)

[6]Jumps could also happen in the liabilities of a company as seen in the case of the steep rise in credit spreads of BP after the oil spill in the Gulf of Mexico. But this too does not appear to be relevant to the banks during the global financial crisis.

The Tenor Swap Spread and CCS Spread

Post crisis, we assume that Libor is a risky rate that includes a credit spread (except possibly at the overnight horizon). Recall also that future Libor rates are not the rate at which a specific Libor rated entity can borrow in future. For example, if Bank X is Libor rated today, it can borrow at Libor today[1], but there is no assurance that it can borrow at Libor next quarter because it may no longer be Libor rated next quarter. The Libor rate next quarter will be the rate at which a bank can borrow if it is Libor rated on that date.

There is therefore a big difference between a six month Libor deposit and a three month Libor rated deposit that is rolled over after three months into another three month Libor rated deposit. The latter is less risky because the deposit is known to be with a Libor rated bank at the beginning of the second quarter while there is no such assurance in the second case. The six month Libor deposit should therefore yield more than the rate obtained by compounding the current three month Libor rate with the expected three month Libor rate next quarter.

This implies that an interest rate swap whose floating leg is three month Libor is not the same as a swap whose floating leg is six month Libor. The floating leg payments on the first swap are expected to be lower and therefore the fixed leg should also be lower. A tenor swap in which both legs are floating – say three month Libor on one leg and six month Libor on the other leg – should include a tenor spread on the three month leg to ensure that the swap is fair at inception. We are assuming here that the interest rate swaps are free of credit (counterparty) risk as discussed later when we turn to collateralization.

Similar explanations can be given for the Cross Currency Swap (CCS) spread. In a CCS, both legs are floating but in different currencies. For example, US dollar Libor on a dollar notional amount may be exchanged for Japanese yen Libor on an equivalent yen notional with an exchange of principals at the end. If Libor is regarded as risk free, then the two legs are floating rate bonds that must be worth par and the swap should trade flat. In reality however, for several years, there has been a premium on the yen leg of this swap. Credit quality differences could be one explanation for this, though liquidity issues might also play an important role[2].

Old Style Forward Rates

Pre-crisis, when Libor is regarded as risk free, forward rates was computed along any curve (say the six month interest rate swap curve) by bootstrapping as follows:

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[1]At the peak of the crisis, even this was perhaps not true leading to the joke that Libor is the rate at which banks do not lend to each other. But outside those truly dark months, it is reasonable to assume that top rated banks can borrow at Libor.

[2]At a fundamental level, it is very difficult if not impossible to distinguish between funding liquidity risk and credit risk.

- The current six month Libor, $L_{6m} \equiv L_{6m}(0)$, can be used to compute the price today of a six month zero coupon bond as $P_{6m} \equiv P_{6m}(0) = \left(1 + \frac{L_{6m}}{2}\right)^{-1}$

- The one year interest rate swap rate based on six month Libor, S_{6m}^{1y}, is the yield of a one year par bond with semi-annual coupons equal to $\frac{S_{6m}^{1y}}{2}$. This par bond has a year one cash flow of $1 + \frac{S_{6m}^{1y}}{2}$ and the present value of this cash flow is $1 - P_{6m}\frac{S_{6m}^{1y}}{2}$ (subtracting the present value of the first coupon from the par value of the bond. The price of a one year zero coupon bond is therefore $P_{1y} \equiv P_{1y}(0) = \frac{1 - P_{6m}\frac{S_{6m}^{1y}}{2}}{1 + \frac{S_{6m}^{1y}}{2}}$.

- The forward rate today for six month Libor six months from now is then given by $F_{6m}^{6m} \equiv F_{6m}^{6m}(0) = 2\left[\frac{P_{6m}}{P_{1y}} - 1\right]$

The above analysis can also be justified in terms of risk neutral valuation because the forward rate[1] computed at time *t* using the above equation

$$F_{6m}^{6m}(t) = 2\left[\frac{P_{6m}(t)}{P_{1y}(t)} - 1\right] = \frac{2P_{6m}(t) - 2P_{1y}(t)}{P_{1y}(t)} \tag{1}$$

is the price at time *t* of the portfolio $2P_{6m}(t) - 2P_{1y}(t)$ relative to the numeraire $P_{1y}(t)$ and is therefore a martingale using the risk neutral probabilities associated with this numeraire. At time $t = \text{six months}$, the right hand side becomes simply the six month Libor prevailing at that date given by $2\left[\frac{1}{P_{1y}(6m)} - 1\right]$. The forward rate is therefore the risk neutral expectation of Libor.

$$F_{6m}^{6m}(t) = E^{\#} 2\left[\frac{1}{P_{1y}(6m)} - 1\right] \tag{2}$$

where $E^{\#}$ denotes the risk neutral expectation with the one year Libor bond as the numeraire.

Alternatively, we can also think of this equation in terms of a replication strategy – to make a forward six month deposit beginning six months from now, we borrow for six months and make a one year deposit right now.

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[1]This formula is the same as Eq 3 of Mercurio (2009).

Under the post crisis assumptions, this methodology is not valid.

- The replication approach does not work because if we make a one year deposit now with a Libor rated bank, there is no assurance that at the end of six months, that bank would remain Libor rated. Hence we do not replicate the forward Libor deposit which by definition is with a Libor rated bank at inception (six months from now).

- The bootstrapping approach is not valid because S_{6m}^{1y} is no longer the yield of a par bond.

- The martingale approach does not work because $P_{1y}(t)$ is no longer the price of a traded asset and cannot therefore be used as a numeraire.

New Style FRA Rates

The new method does not use the bootstrapped forward rate, but relies instead on the Forward Rate Agreement (FRA) rate – the rate quoted in the market for a forward Libor deposit. Under risk neutral valuation, the FRA rate can also be computed as the expectation of the Libor rate using the OIS rate as the risk free discount rate.

When we take OIS as the risk free rate, we are implicitly treating the excess of the Libor rate (or the swap rate) over the OIS rate for the same maturity as the expected default loss. For example, assume for simplicity that the recovery from a defaulting bank is zero and that Q^* is the risk neutral probability that a bank that is Libor rated bank six months from today will not default during the six months thereafter. The FRA rate FRA_{6m}^{6m} for a six month Libor deposit beginning six months from today is fair if the expected present value of the deposit six months is equal to the the expected present value of the repayment after one year. Letting P^{OIS} denote the prices of zero coupon bonds based on the OIS curve, we get[1]

$$P_{6m}^{OIS} = P_{1y}^{OIS}\left(1+\frac{1}{2}FRA_{6m}^{6m}\right)Q^* \quad \text{or} \quad FRA_{6m}^{6m} = 2\left[\frac{P_{6m}^{OIS}}{P_{1y}^{OIS}Q^*}-1\right] \tag{3}$$

It is seen that the FRA rate equals the bootstrapped forward rate if there is no default in which case Q^* is equal to unity and the OIS curves and Libor curves are the same.

The difference between the FRA rate and the forward rates can be illustrated by the following example. Suppose the price of the 6m and 12m zero coupon OIS bonds are 0.980392 and 0.956938 corresponding to simple interest rates of 4.0% and 4.5%. The OIS forward rate is seen to be $2\left(\frac{0.980392}{0.956938}-1\right)=4.90\%$.

Suppose that during any six month period, three things can happen to a Libor rated bank:

[1]This formula is the same as that in Table 1 of Mercurio (2009).

- It defaults with risk neutral probability 0.5% with zero recovery.
- It gets downgraded with risk neutral probability 5.0%. After downgrade, it defaults with probability risk neutral 4% during the subsequent six months with zero recovery.
- It remains a Libor rated bank with risk neutral probability 94.5%.

This implies that the six month risk neutral survival probability of a Libor rated bank is $1-0.5\%=99.5\%$ and the twelve month neutral survival probability is $94.5\% \text{ x } 99.5\%+5.0\% \text{ x } 96.0\%=94.03\%+4.80\%=98.83\%$. These imply rather high default probabilities and I am using these admittedly unrealistic assumptions only to highlight the phenomenon more clearly

By risk neutral valuation, six month Libor should be 5.0251% and twelve month Libor should be 5.7398% as shown below:

$$\left(1+\frac{5.0251\%}{2}\right) \text{ x } 0.980392 \text{ x } 99.50\%=1.0$$

$$(1+5.7398\%) \text{ x } 0.956938 \text{ x } 98.83\%=1.0$$

The Libor forward rate is then seen to be 6.30% as shown below:

$$2\left(\frac{1+5.7398\%}{1+5.0251\%/2}-1\right)=6.30\%$$

However, the FRA rate is 5.93% as computed below using Eq (3):

$$2\left(\frac{0.980392}{0.956938 \text{ x } 99.5\%}-1\right)=5.93\%$$

The FRA rate is different from both the OIS forward rate and the Libor forward rate. The former difference is simply a reflection of the higher default risk of the Libor. The latter difference is more subtle – it reflects the possibility of the Libor rated bank being downgraded in the first six months and then defaulting at much higher rates thereafter.

The twelve month Libor includes a 0.20% probability of default in the second six months following a downgrade in the first six months ($5\% \text{ x } 4\%=0.20\%$). The forward Libor computed from six month and twelve month Libor includes the effect of this default probability. The FRA rate does not take this into account because the FRA does not contemplate lending to a downgraded bank. This 20 basis points of extra default probability during six months (40 basis points annualized) is what causes the forward Libor rate to be 37 basis points more than the FRA rate. Needless to say, this is an extreme example with unrealistically high default and downgrade probabilities. In normal market situations, we would expect rather smaller differences.

Evidently, it is possible to run this exercise in reverse to extract default probabilities from OIS, Libor and FRA rates.[1] An alternative and perhaps more convenient approach is to compute the FRA rate directly using risk neutral valuation[2]:

$$FRA_{6m}^{6m} = 2\, E^* \left[\frac{1}{P_{1y}(6m)} - 1 \right] \tag{4}$$

where E^* denotes the risk neutral expectation using the OIS as the risk free rate. Comparing the above equation for the FRA rate with Eq (2) for the (old model) Libor forward rate, we see that we are taking the expectation of the same quantity but using a different set of risk neutral probabilities. Essentially, we are moving from regarding the Libor/swap rate as risk free to regarding the OIS as risk free.

This implies that the difference between the Libor forward rate and the Libor FRA rate is the familiar quanto adjustment that arises in any change of measure:

$$\begin{aligned} &FRA_{6m}^{6m} = QA \text{ x } F_{6m}^{6m} \\ &\text{where } QA = \exp\left(-\int_0^{6m} \sigma_F \sigma_X \rho_{F,X} \right) \\ &\text{and the numeraire ratio } X = \frac{P_{1Y}(t)}{P_{1Y}^{OIS}(t)} \end{aligned} \tag{5}$$

If we take the correlation ρ to be equal to 1 for simplicity and assume that the volatility of forward Libor is 30% and the volatility of the numeraire ratio is 15%, then $\sigma_F \sigma_X \rho_{F,X} = 30\% \text{ x } 15\% \text{ x } 1.0 = 4.5\%$ and integrating this for six months and exponentiating gives $QA = \exp(-2.25\%) = 0.9778$. Under these assumptions, a forward rate of 6.30% would imply an FRA rate of $0.9778 \text{ x } 6.30\% = 6.20\%$. This difference of 10 basis points is much less than the 37 basis points that we obtained under extreme assumptions regarding default rates and recovery.

Two curve mechanics

The two curve discounting process can be summarized as follows:

- Bootstrap[3] the OIS curve to get zero rates for various maturities.
- Value all instruments by using risk neutral valuation with the OIS rates as the discount rates.

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[1]This formula of course requires the assumption of zero recovery. In case of non zero recovery, the formula would need to be modified.

[2]This formula is the same as the unnumbered equation just above Eq 5 of Mercurio (2009).

[3]In practice, it would be necessary to use splines or some other interpolation scheme to perform the bootstrap.

- When cash flows that depend on future Libor, the risk neutral expectations are based on the OIS rate as the numeraire. Instead of the old model assumption that forward Libor rates are realized, we assume that FRA rates are realized.

To value caps and swaptions, the following modification of the standard Libor Market Model (LMM) procedure can be used:

- For each Libor curve (1 month, 3 month, 6 month and so on), compute the FRA rates as the risk neutral expectation of the forward rates with the OIS rates as the risk free rate.

- Assume that the FRAs are lognormally distributed.

- Value Libor caps and swaptions using the standard LMM with only one change – use the FRA rates instead of the forward rates obtained by bootstrapping the Libor curves.

Collateralization

It only remains to justify the use of the OIS rate as the risk free rate. After all the opaqueness of bank balance sheets implies that the "jump to default" risk cannot be ignored at even the one day horizon. The risk has to be further attenuated to justify ignoring the risk.

In the post crisis environment, most derivative transactions between even highly rated banks tend to be collateralized on a daily basis. The risk in these swaps is extremely low for two reasons:

1. At an overnight horizon, the risk of a Libor rated bank defaulting is quite low even with all the opaqueness of bank balance sheets. Even during the worst phase of the crisis in 2008, it took several days or even weeks for a bank to move from apparent health to serious distress.

2. In a swap with daily collateral flows, the exposure of one bank to another is not the notional value of the swap but only the potential change in its market value over the course of one day. This exposure is only a minuscule fraction of the notional value of the swap.

The effective risk is the product of two small risks and can be regarded as negligible even when neither of these two small risks is negligible by itself. For modelling purposes, we look at a formulation in which the collateralization happens on a continuous basis and the risk is theoretically zero.

The daily collateralized swap is regarded as a very close approximation to this ideal situation and the risk though non zero may be regarded as negligible. In this formulation, it can be shown that the discount rate should be the rate paid on the collateral that is exchanged between the banks.

To understand this result, it is useful to consider the difference between forward contracts, futures contracts, and collateralized forward contracts.

Forward Contracts

To analyse forward contracts, we use the zero coupon bond as the numeraire, let $S(t)$ denote the price of the underlying and let $F(t)$ be the value of a forward contract on *S* at the forward price *f* prevailing at time 0. By risk neutral valuation, we have

$$\begin{aligned} &F(t)=P(t,T)E^{*}[F(T)] \\ &=P(t,T)E^{*}[S(T)-f] \end{aligned} \tag{6}$$

The discount factor is the price of the zero coupon bond maturing at expiry of the forward contract and this bond is also the numeraire that determines the risk neutral probabilities. With this numeraire and the associated risk neutral probabilities, the forward price at inception (time 0) is the expected price of the underlying:

$$\begin{aligned} &0=F(0)=P(0,T)E^{*}[S(T)-f] \\ &\Rightarrow \\ &E^{*}[S(T)]=f \end{aligned} \tag{7}$$

Futures Contracts

To analyse the futures contract, we change the numeraire to the money market account[1] and consider a trading strategy that starts at time 0, ends at time *T*, and holds $e^{\sum_0^t r_i \Delta}$ futures contracts at time *t* ($0 \le t \le T$) and invests all mark to market gains and losses into a money market account. This is the inverse of the tailing of the hedge strategy where we start with less than one futures contracts and build the position up gradually to end with one futures contract. Here we start with one futures contract and let it grow gradually to a random number of contracts at maturity.

Since the futures contracts are always worth zero, the value of this strategy is the value of the money market account. At time *T*, this contains all mark to market cash flows plus accumulated interest on each of these cash flows.

We derive the expression for the value M_i of the money market account as follows. Consider the mark to market cash flow at time *t*: $\left[e^{\sum_0^t r_i \Delta}\right](F_t - F_{t-1})$. Together with future accumulated interest up to time *T*, this becomes $\left[e^{\sum_0^t r_i \Delta}\right]\left[e^{\sum_t^T r_i \Delta}\right](F_t - F_{t-1}) = \left[e^{\sum_0^T r_i \Delta}\right](F_t - F_{t-1}) = B[0,T](F_t - F_{t-1})$ where $B[0,T]$ is the accumulated value at time *T* of a money market account starts with unit value at time 0. Adding all these accumulated values in this telescoping sum, we get the

[1]Here $\Delta = 1/365$. For simplicity, we are assuming that the futures contract is marked to market daily and the money market account is reinvested every day at the then prevailing overnight rate. The same analysis could be carried out using continuous mark to market and the continuous money market account.

terminal value of the money market account $M(T)$ as $B(0,T)[F(T)-F(0)]$ or $B(0,T)[S(T)-F(0)]$ since $F(T)=S(T)$. Noting that $M(0)=0$ and applying risk neutral valuation to M_t gives us:

$$0=M_0=E^*\left[\frac{1}{B(0,T)}B(0,T)[S_T-F_0]\right]=E^*[S_T-F_0] \tag{8}$$
$$\Rightarrow$$
$$F_0=E^*[S_T]$$

So the future price is the expected price of the underlying under the martingale measure if the money market account is the numeraire.

The difference between the forward and futures price comes from the difference between the two martingale measures.

Collaterized Forward Rates[1]

Let r_i be the risk free rate on day i and c_i be the collateral rate – the rate that the receiving bank pays on the cash collateral posted with it. Consider a trading strategy that starts at time 0, ends at time *T*, and holds $e^{\sum_0^t y_i \Delta}$ collateralized forward contracts at time *t* ($0 \le t \le T$) where $y_i = r_i - c_i$. Let the value of the collateralized forward contract at time *i* be given by h_i.

Consider the mark to market cash flow at time *t*: $\left[e^{\sum_0^t y_i \Delta}\right](h_t - h_{t-1})$. The receiving bank can invest this cash flow at the risk free rate, but must pay the collateral rate on it. Its net return on this cash flow is therefore y_i. Together with future accumulated net return up to time *T*, this becomes $\left[e^{\sum_0^t y_i \Delta}\right]\left[e^{\sum_t^T y_i \Delta}\right](h_t - h_{t-1}) = \left[e^{\sum_0^T y_i \Delta}\right](h_t - h_{t-1}) = C[0,T](h_t - h_{t-1})$ where $C[0,T]$ is the accumulated value at time *T* of a collateral account starts with unit value at time 0. consider a collateral account that starts with h_0 at time 0 and into which all the mark to market cash flows are added along with the net return Adding all these accumulated values in this telescoping sum, we get the terminal value of the collateral account V_T as $V_T = C(0,T)[h_0 + (h_T - h_0)] = C(0,T) h_T$. Discounting this at the overnight risk free rate, and taking risk neutral expectations (with the money market account as the numeraire), we get:

$$h_0 = E^*\left[e^{\sum_0^t -r_i \Delta} C(0,T) h_T\right] = E^*\left[e^{\sum_0^t -r_i \Delta} e^{\sum_0^t y_i \Delta} h_T\right] = E^*\left[e^{\sum_0^t c_i \Delta} h_T\right] \tag{9}$$

[1]The discussion in this section closely follows section 3.1 of Masaaki Fujii, Yasufumi Shimada, and Akihiko Takahashi (2010a).

The risk free rate completely drops out of the valuation formula and we obtain the result that discounting must be done at the collateral rate even if it is not the true risk free rate. The collateralization does not however change the numeraire which remains the money market account which earns the risk free rate.

For example, we may believe that the overnight US Treasury repo rate in the US is the true risk free rate and the OIS rate may be above the repo rate because of the credit risk of even overnight lending. That does not invalidate discounting at the OIS rate if the swap is collateralized. However, the numeraire is not the collateral account; it is the money market account.

In some sense, we could regard this as a three curve model – the repo rate providing the numeraire, the OIS rate providing the discount rate, and the cash flows are determined by Libor (if the underlying of the collateralized forward contract is Libor).

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