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# Giffen goods in a mean variance framework

This note demonstrates (in the context of two uncorrelated assets) that the optimal allocation to a safe asset cannot increase if its risk increases.

Let $\mu$ be the mean vector and $\Sigma$ the variance covariance matrix. Let $\underline{1}$ be a vector of ones and $A$ be the risk aversion coefficient. The mean variance optimization problem is

$$\begin{aligned}&\max\ \mu' w - \frac{1}{2} A w' \Sigma w - \lambda w' 1\\ &\text{s.t. } w' \underline{1} = 1\end{aligned}$$

The Lagrangian for the constrained maximization problem is:

$$\mu' w - \frac{1}{2} A w' \Sigma w - \lambda w' 1$$

The first order condition says:

$\mu - A' \Sigma w - \lambda \underline{1} = 0$ or

$$w = \frac{1}{A}\left(\Sigma^{-1}\mu - \lambda \Sigma^{-1} 1\right) = \frac{1}{A} e - \frac{\lambda}{A} p$$

where $e = \Sigma^{-1}\mu$ and $p = \Sigma^{-1}\underline{1}$ .

To compute $\lambda$, we proceed as follows:

$$\begin{aligned}1 &= 1' w = \frac{1}{A} 1' e - \frac{\lambda}{A} 1' p\\ A &= \underline{1}' e - \lambda \underline{1}' p\\ \lambda &= \frac{\underline{1}' e - A}{1' p}\end{aligned}$$

$$\begin{aligned}w &= \frac{1}{A} e - \frac{\underline{1}' e}{A \underline{1}' p} p + \frac{1}{\underline{1}' p} p\\ &= \frac{1}{\underline{1}' p} p + \frac{1}{A}\left(e - \frac{\underline{1}' e}{\underline{1}' p} p\right)\\ &= \frac{1}{\underline{1}' \Sigma^{-1} \underline{1}} \Sigma^{-1} 1 + \frac{1}{A} \Sigma^{-1} \mu^*\\ \mu^* &= \mu - \frac{\underline{1}' \Sigma^{-1} \mu}{\underline{1}' \Sigma^{-1} \underline{1}} \underline{1}\end{aligned}$$

The optimal portfolio can thus be written as

$$w = MVP + \frac{1}{A} SP$$

The minimum variance portfolio $MVP = \frac{1}{\underline{1}' \Sigma^{-1} \underline{1}} \Sigma^{-1} \underline{1}$

The zero investment purely speculative portfolio $SP = \frac{1}{A} \Sigma^{-1} \mu^*$

$$\mu^* = \mu - \frac{\underline{1}' \Sigma^{-1} \mu}{1' \Sigma^{-1} 1} 1$$

We now consider two uncorrelated assets with variances

$$\sigma_1^2 = \frac{1}{h_1}, \sigma_2^2 = \frac{1}{h_2}$$

In this special case, the optimal portfolio takes a simple form as shown below:

$$\Sigma^{-1} = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$$

$$MVP = \frac{1}{h_1 + h_2} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$\mu^* = \mu - \frac{h_1 \mu_1 + h_2 \mu_2}{h_1 + h_2} 1 = \frac{(\mu_1 - \mu_2)}{h_1 + h_2} \begin{bmatrix} h_2 \\ -h_1 \end{bmatrix}$$

$$SP = \frac{1}{A} \frac{(\mu_1 - \mu_2)}{h_1 + h_2} \begin{bmatrix} h_1 h_2 \\ -h_1 h_2 \end{bmatrix}$$

$$w = \frac{1}{h_1 + h_2} \begin{bmatrix} h_1 \left( 1 + h_2 \frac{(\mu_1 - \mu_2)}{A} \right) \\ h_2 \left( 1 - h_1 \frac{(\mu_1 - \mu_2)}{A} \right) \end{bmatrix}$$

We assume that the first asset is the safer asset with a lower risk and lower return as well. Its lower risk means that it has the larger weight (>0.5) in the minimum variance portfolio. But its lower return implies that it has a negative weight in the speculative portfolio. At sufficiently high levels of risk aversion, the minimum variance portfolio dominates and the investor is net long the safe asset. We note that this situation $(w_1 > 0)$ obtains if and only if $A > h_2(\mu_2 - \mu_1)$

To see how the weight in the safer asset varies with its risk, we differentiate to get:

$$\frac{d w_1}{d h_1} = \frac{h_2}{(h_1 + h_2)^2} \left[ 1 + \frac{h_2(\mu_1 - \mu_2)}{A} \right]$$

This derivative is positive if and only if $A > h_2(\mu_2 - \mu_1)$ . This means that $w_1$ is increasing with $h_1$ (decreasing with $\sigma_1$ ) if and only if $w_1$ is positive.

If the investor is net long the safer asset, then an increase in its risk will cause the investor to sell the safer asset and reduce his holding of this asset. But if the investor is net short the safer asset, then an increase in its risk will cause the investor to buy the asset and reduce his short position in this asset.

What is happening is that the weight in the minimum variance portfolio always declines with increasing risk. But the greater risk makes it more risky to short the asset and therefore the magnitude of the short position in the speculative portfolio also declines. Which effect dominates depends on which position (the long position in the minimum variance portfolio or the short position in the speculative portfolio) was larger.